MATHEMATICAL ANALYSIS FOR RESERVOIR MODELS

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Abstract. In the first part of this paper, the mathematical analysis is presented in detail for the single-phase, miscible displacement of one fluid by another in a porous medium. It is shown that initial boundary value problems with various boundary conditions for this miscible displacement possess a weak solution under physically reasonable hypotheses on the data. Then in the second part of this paper, it is proven how the analysis can be extended to two-phase fluid flow and transport equations in a porous medium. The flow equations are written in a fractional flow formulation so that a degenerate elliptic-parabolic partial differential system is produced for a global pressure and a saturation. This degenerate system is coupled to a parabolic transport equation which models the concentration of one of the fluids. The analysis here does not utilize any regularized problem; a weak solution is obtained as a limit of solutions to discrete time problems.

Key words. porous medium, flow and transport, elliptic-parabolic system, degenerate equations, existence

AMS subject classifications. 35K60, 35K65, 76S05, 76T05

1. Introduction.

Multiphase flow and transport of fluids in porous media is of importance socially and economically in a number of applications. Petroleum engineers have been interested in efficient recovery of energy resources, and hydrologists have been concerned with improvement of groundwater resource utilization for a long time, for example. Unfortunately, despite the great progress made in development of physical models of multiphase flow and transport of fluids in porous media, there has been limited mathematical theory behind these models. The difficulty stems from the fact that the equations modeling complex physical phenomena involving both flow and transport of fluids are often coupled systems of nonlinear, time-dependent degenerate partial differential equations. Hence simplied models have been dealt with in the last twenty years.

The simplest porous media problem corresponds to the flow of a fluid where a whole porous medium is filled with the single fluid (usually gas or oil in petroleum engineering, or water in groundwater hydrology). The usual equations for the single-phase flow model are of parabolic type for the fluid density or pressure, and are well understood (see, e.g., [5, 8]).

A more complex case involves the single-phase, miscible displacement of one fluid by another in a porous medium. Under the assumption that no volume change results from the mixing of the two fluids, a coupled, nonlinear differential system of two equations is often utilized for this miscible displacement problem. One of the equations is of elliptic (respectively, parabolic) type for the fluid pressure if the fluids are incompressible (respectively, compressible), and the other is of parabolic type for the concentration of one fluid. This system is complicated by the facts that the pressure equation can be degenerate due to the form of the concentration-dependent viscosity and that the transport and diffusion/dispersion coefficients in the concentration equation can be unbounded due to the potentially unbounded fluid velocity.

The miscible displacement problem was first studied by Sammon [20] where a one-dimensional model was theoretically analyzed, and the viscosity of the fluid was

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assumed to be independent of the concentration. The latter assumption decouples the pressure equation from the concentration equation. Mikelić [18] later analyzed a three-dimensional stationary displacement problem. While the viscosity was allowed to be concentration-dependent, it was in fact assumed to be sufficiently close to a constant. Then the pressure and concentration equations were essentially separated so that well-posedness for the stationary problem can be established. Recently, Feng [14] considered a two-dimensional model for the displacement problem. The viscosity can be concentration-dependent, but the analysis was valid only for a two-dimensional problem. Furthermore, the analysis, following Kružkov and Sukorjanskiĭ [16] for two-dimensional two-phase immiscible flow, made use of the corresponding regularized system and required the coefficients of the regularized problem to be bounded uniformly with respect to the regularization parameter. The uniform boundedness is hardly satisfied due to the above mentioned feature for the transport and diffusion/dispersion coefficients. Finally, all the theoretical results in [20, 18, 14] were obtained solely for homogeneous Newmann boundary conditions and without gravity effects. The study of the single-phase, miscible displacement of one incompressible fluid by another from the numerical point of view using finite element methods has been extensively carried out (see, e.g., [12]); the case where the components are assumed to be slightly compressible has been also numerically studied (see, e.g., [13]).

In this paper the miscible displacement of one incompressible fluid by another in a porous medium is further investigated. A time-dependent, three-dimensional displacement problem with various boundary conditions and gravity effects, including mixed nonhomogeneous ones, is shown to possess a weak solution. The viscosity can be concentration-dependent, and the assumptions required on the data in the earlier papers are weakened; only physically reasonable assumptions are made. The analysis makes no use of the corresponding regularized problem; a weak solution is obtained as a limit of solutions to discrete time problems. It follows Alt and Luckhaus [3] for treating quasilinear elliptic-parabolic differential equations. The technique was later exploited by Alt-di Benedetto [2] and Arbogast [6] for handling the two-phase immiscible flow problem. However, we point out that the miscible displacement problem is different from the two-phase immiscible problem due to above mentioned difficulties. In particular, in the present problem special care must be taken on the transport and diffusion/dispersion coefficients in the concentration equation. We here introduce a solution-dependent space to handle this difficulty.

The above two cases deal with single-phase flow. Two-phase flow is more complex and is of greater practical interest. This case corresponds to the so-called secondary recovery in petroleum reservoirs where two fluid phases (usually water and oil) flow simultaneously, or to the fluid movement in an air-water porous media system in groundwater hydrology. In the last two decades, a considerable amount of effort has been made solely in the analysis of flow equations of two-phase incompressible, immiscible type. Transport equations have not been handled for the two-phase system. Existence of weak solutions for the flow equations has been established under various assumptions on physical data (see, e.g., [5, 16, 8, 15, 2, 6]). Numerical analyses of the flow equations of compressible type using finite elements have been carried out in [9, 10].

The second part of this paper is to extend the analysis for the single-phase, miscible displacement of one fluid by another to a two-phase flow and transport system. In addition to a strong coupling between flow and transport equations, the whole differential system combines the above difficulties for the displacement problem with those for the flow equations. In particular, the flow equations are usually degenerate, the number of equations is not a-priori known at a given place of a porous media, and the capillary pressure function is generally unbounded. Here we make an initial attempt to analyze both the flow and transport equations for a two-phase system using the techniques for the single-phase, miscible displacement problem.

In the next section we consider the single-phase displacement problem. We begin with what is meant by a weak solution. Then we carefully state the assumptions on the physical data required for the main result obtained in this part. Most of this section is devoted to the proof of the main result. In §3, we extend the results to a two-phase flow system. We shall follow the usual practice [4, 8] to write the flow equations of this system in a fractional flow formulation, i.e., in terms of a saturation and a global pressure so that the elliptic part of the system for this global pressure and the parabolic part for the saturation are separated. The concentration equation is obtained from the usual conservation law.

We end with a remark that uniqueness of the weak solution remains open. This is due to the coupling between the partial differential equations under consideration, which makes it difficult to obtain enough regularity of the solution. When the solution is assumed to have enough regularity (e.g., in the semi-classical sense), the uniqueness can be shown in the usual way [17].

2. Miscible Displacement of One Fluid by Another. In §2.1 the differential system for the single-phase, miscible displacement of one incompressible fluid by another in a porous medium $\Omega \subset \Re^d$ $(d \leq 3)$ is described. Then in §2.2 we state the assumptions on the physical data, define what is meant by a weak solution, and state the main result shown in this section. The proof of the main result is presented in §2.3, and two of the lemmas needed for the main result are proven in §2.4.

2.1. The differential system. The usual equations describing two-component, incompressible, miscible displacement are given by (see, e.g., [7, 19])

$$\begin{aligned} &-\nabla \cdot \{k(x)(\nabla p - \rho g)/\mu(c)\} = q^I - q^P, \\ &\phi(x)\partial_t c - \nabla \cdot \big(D(u)\nabla c\big) + u \cdot \nabla c + q^I c = \hat{c} q^I, \end{aligned} \tag{2.1}$$

for $(x,t) \in \Omega_T \equiv \Omega \times J$ with $J = (0,T]$ $(T > 0)$, where ϕ and k are the porosity and absolute permeability of the porous medium, μ and ρ are the viscosity and density of the fluid mixture, g denotes the gravitational, downward-pointing, constant vector, c indicates the concentration of one of the two components, p is the pressure of the fluid, D is the diffusion/dispersion coefficient, $\hat{c}$ is the injected concentration, q^I and q^P represent the sum of injection well source terms and production well sink terms, respectively, and u is the Darcy velocity of the fluid defined by

$$u = -\frac{k(x)}{\mu(c)}(\nabla p - \rho g). \tag{2.2}$$

For $\Gamma = \partial\Omega$, let

$$\Gamma = \Gamma_1^p \cup \Gamma_2^p = \Gamma_1^c \cup \Gamma_2^c, \quad \Gamma_1^p \cap \Gamma_2^p = \Gamma_1^c \cap \Gamma_2^c = \emptyset.$$

With this division of Γ, the boundary conditions are specified by

$$\begin{aligned} &u \cdot \nu - a_1(c)p = \varphi_1(x,t), && (x,t) \in \Gamma_1^p \times J, \\ &p = \varphi_2(x,t), && (x,t) \in \Gamma_2^p \times J, \\ &-(D\nabla c) \cdot \nu - a_2(c)c = \varphi_3(x,t), && (x,t) \in \Gamma_1^c \times J, \\ &c = \varphi_4(x,t), && (x,t) \in \Gamma_2^c \times J, \end{aligned} \tag{2.3}$$

where the a_i and φ_j are given functions ($i = 1, 2,\ 1 \le j \le 4$), and ν is the outward unit norm to Γ. The initial condition is given by

$$c(x, 0) = c_0(x), \quad x \in \Omega. \tag{2.4}$$

The differential system given by (2.1)–(2.4) for the main unknowns p and c will be studied in this section.

2.2. Assumptions and the main result. The usual Sobolev spaces $W^{l,\pi}(\Omega)$ with the norm $\|\cdot\|_{W^{l,\pi}(\Omega)}$ ([1]) will be used, where l is a nonnegative integer and $0 \le \pi \le \infty$. When $\pi = 2$, we simply write $H^l(\Omega) = W^{l,2}(\Omega)$. When $l = 0$, we have $L^2(\Omega) = H^0(\Omega)$. Below $(\cdot,\cdot)_Q$ denotes the $L^2(Q)$ inner product (Q is omitted if $Q = \Omega$). We now make the following assumptions:

(A1) $\Omega \subset \Re^d$ is a multiply-connected domain with Lipschitz boundary Γ, $\Gamma = \Gamma_1^p \cup \Gamma_2^p = \Gamma_1^c \cup \Gamma_2^c$, $\Gamma_1^p \cap \Gamma_2^p = \Gamma_1^c \cap \Gamma_2^c = \emptyset$, and each Γ_i^p and Γ_i^c is a $(d-1)$-dimensional domain.

(A2) $\phi \in L^\infty(\Omega)$, $\phi(x) \ge \phi_* > 0$, and $k(x)$ is a bounded, symmetric, and uniformly positive definite matrix, i.e.,

$$0 < k_* \le |\xi|^{-2} \sum_{i,j=1}^{d} k_{ij}(x)\xi_i\xi_j \le k^* < \infty, \quad x \in \Omega,\ \xi \ne 0 \in \Re^d.$$

(A3) The diffusion/dispersion term is given by

$$D(u) = \phi\{d_{mo}I + |u|\big(d_l E(u) + d_t E^-(u)\big)\},$$

where I is the d-by-d identity matrix, $d_{mo} > 0$ is the molecular diffusion coefficient, d_l and d_t are the longitudinal and transverse dispersion coefficients, respectively, the matrix $E(u)$ is the projection along the direction of flow determined by

$$E(u) = \left(\frac{u_i u_j}{|u|^2}\right), \quad |u| = \sqrt{u_1^2 + u_2^2}, \quad u = (u_1, u_2),$$

and $E^-(u) = I - E(u)$.

(A4) The following form is being widely used for the viscosity μ:

$$\mu(c) = \mu(0)\big(1 + (\mathcal{M}^{1/4} - 1)c\big)^{-4} \quad \text{for } c \in [0, 1],$$

where $\mathcal{M} = \mu(0)/\mu(1)$ is the mobility ratio.

(A5) q^I, $q^P \ge 0$, $q^I \in L^\infty(J; L^2(\Omega))$, and $q^P \in L^\infty(J; H^{-1}(\Omega))$.

(A6) In the case of $\Gamma_2^p = \emptyset$ and $a_1 \equiv 0$,

$$\int_{\Gamma_1^p} \varphi_1 \mathrm{d}\sigma = \int_\Omega (q^I - q^P)\mathrm{d}x.$$

(A7) There is a subset $\Gamma_{1,*}^p \subset \Gamma_1^p$ (with nonzero measure only if $\Gamma_2^p = \emptyset$ and $a_1 \not\equiv 0$) such that $a_1 \ge a_{1,*} > 0$ on $\Gamma_{1,*}^p \times J \times [0, 1]$.

(A8) The boundary data satisfy

$$\begin{aligned}
&\varphi_1 \in L^\infty(J; H^{-1/2}(\Gamma_1^p)), \quad \varphi_3 \in L^2(J; H^{-1/2}(\Gamma_1^c)),\\
&\varphi_2 \in L^\infty(J; H^1(\Omega)), \quad \varphi_4 \in L^2(J; W^{1,4}(\Omega)),\\
&\partial_t \varphi_4 \in L^1(\Omega_T), \quad 0 \le \varphi_4(x,t) \le 1 \quad \text{a.e. on } \Omega_T.
\end{aligned}$$

(A9) $a_1,\ a_2 \geq 0$, and the norms $|||a_1|||_{L^\infty(\Omega_T)}$ and $|||a_2|||_{L^\infty(\Omega_T)}$ are bounded, where

$$|||v||| = \Big\| \sup_{c\in[0,1]} |v(x,c)| \Big\|,$$

for any given norm.

(A10) $\hat{c}$ and c_0 satisfy $0 \leq \hat{c} \leq 1$ a.e. on Ω_T and $0 \leq c_0 \leq 1$ a.e. on Ω.

We introduce the spaces

$$\begin{aligned} V &= \{v \in H^1(\Omega) : v|_{\Gamma_2^p} = 0;\ \text{if } \Gamma_2^p = \emptyset \text{ and } a_1 \equiv 0,\ \text{then } \textstyle\int_\Omega v \mathrm{d}x = 0\}, \\ W &= \{v \in H^1(\Omega) : v|_{\Gamma_2^c} = 0\}. \end{aligned}$$

Below V^* and W^* indicate the duals of V and W, respectively.

Definition 2.1. A weak solution of the system in (2.1)–(2.4) is a pair of functions (p, c) with $p \in L^\infty(J;V) + \varphi_2$, $c \in L^2(J;W(u)) + \varphi_4$ such that

$$\phi\partial_t c \in L^2(J;W^*(u)), \tag{2.5}$$

$$0 \leq c(x,t) \leq 1 \quad \text{a.e. on } \Omega_T, \tag{2.6}$$

$$\begin{aligned} &(a(c)\{\nabla p - \rho g\}, \nabla v) + (a_1(c)p, v)_{\Gamma_1^p} \\ &\quad = (q^I - q^P, v) - (\varphi_1, v)_{\Gamma_1^p}, \quad \forall v \in L^\infty(J;V), \end{aligned} \tag{2.7}$$

$$\begin{aligned} &\textstyle\int_J \langle \phi\partial_t c, v\rangle \,\mathrm{d}t + \int_J (D(u)\nabla c, \nabla v)\mathrm{d}t + \int_J (u\cdot\nabla c, v)\mathrm{d}t \\ &\quad + \textstyle\int_J (q^I c, v)\mathrm{d}t + \int_J (a_2(c)c, v)_{\Gamma_1^c}\mathrm{d}t \\ &\qquad = \textstyle\int_J (\hat{c}q^I, v)\mathrm{d}t - \int_J (\varphi_3, v)_{\Gamma_1^c}\mathrm{d}t, \quad \forall v \in L^2(J;W(u)), \end{aligned} \tag{2.8}$$

$$\begin{aligned} &\textstyle\int_J \langle \phi\partial_t c, v\rangle \,\mathrm{d}t + \int_J (\phi(c-c_0), \partial_t v)\mathrm{d}t = 0, \\ &\qquad \forall v \in L^2(J;W(u)) \cap W^{1,1}(J;L^1(\Omega)), \quad v(x,T) = 0, \end{aligned} \tag{2.9}$$

where $a(c) = k(x)/\mu(c)$, u is given as in (2.2), and the space $W(u)$ is defined by

$$W(u) = \{v \in W : (D(u)\nabla v, \nabla v) < \infty\}.$$

We now state the main result obtained in this section.

Theorem 2.1. *Under assumptions* (A1)–(A10)*, the system in* (2.1)–(2.4) *has a weak solution in the sense of Definition* 2.1.

2.3. Proof of the main result. In this subsection we shall prove Theorem 2.1. We first state the following trivial lemma.

Lemma 2.2. *It holds that*

$$\begin{aligned} d_{mo} + \min(d_l, d_t)|u| \leq &\phi^{-1}|\xi|^{-2} \textstyle\sum_{i,j=1}^d D_{ij}(u)\xi_i\xi_j \\ &\leq d_{mo} + \max(d_l, d_t)|u|, \quad \xi \neq 0 \in \Re^d. \end{aligned}$$

For each positive integer M, divide J into $m = 2^M$ subintervals of equal length $h = T/m = 2^{-M}T$. Set $t_i = ih$ and $J_i = (t_{i-1}, t_i]$ for an integer i, $1 \le i \le m$. Denote the time difference operator by

$$\partial^\eta v(t) = \frac{v(t+\eta) - v(t)}{\eta},$$

for any function $v(t)$ and constant $\eta \in \Re$. Also, for any Hilbert space $\mathcal{H}$, define

$$l_h(\mathcal{H}) = \{v \in L^\infty(J; \mathcal{H}) : v \text{ is constant in time on each subinterval } J_i \subset J\}.$$

For $v^h \in l_h(\mathcal{H})$, set $v^i = v^h|_{J_i}$ for notational convenience. Finally, let

$$\varphi_j^h(x,t) = \frac{1}{h}\int_{J_i} \varphi_j(x,\tau)\mathrm{d}\tau, \quad t \in J_i,\ j = 2,4.$$

Now, the discrete time solution is a pair of functions $p^h \in l_h(V) + \varphi_2^h$, $c^h \in l_h(W) + \varphi_4^h$ satisfing

$$\begin{aligned} &(a(c^h)\{\nabla p^h - \rho g\}, \nabla v) + (a_1(c^h)p^h, v)_{\Gamma_1^p} \\ &\quad = (q^I - q^P, v) - (\varphi_1, v)_{\Gamma_1^p}, \quad \forall v \in l_h(V), \end{aligned} \tag{2.10}$$

and

$$\begin{aligned} &\textstyle\int_J (\phi\partial^{-h}c^h, v)\mathrm{d}t + \int_J (D(u^h)\nabla c^h, \nabla v)\mathrm{d}t + \int_J (u^h \cdot \nabla c^h, v)\mathrm{d}t \\ &\quad \textstyle + \int_J (q^I c^h, v)\mathrm{d}t + \int_J (a_2(c^h)c^h, v)_{\Gamma_1^c}\mathrm{d}t \\ &\quad \textstyle = \int_J (\hat{c}q^I, v)\mathrm{d}t - \int_J (\varphi_3, v)_{\Gamma_1^c}\mathrm{d}t, \quad \forall v \in l_h(W), \end{aligned} \tag{2.11}$$

with

$$u^h = -a(c^h)(\nabla p^h - \rho g). \tag{2.12}$$

This approximate scheme is extended such that $c^h = c_0$ for $t < 0$.

Below C (with or without a subscript) indicates a generic constant independent of h, which will probably take on different values in different occurrences.

Lemma 2.3. *For $h > 0$ small enough, the discrete scheme has a solution such that*

$$0 \le c^h(x,t) \le 1 \quad a.e.\ on \quad \Omega_T. \tag{2.13}$$

The proof of this lemma will be given in the next subsection.

Lemma 2.4. *The solution to the discrete scheme also satisfies*

$$\|p^h\|_{L^\infty(J;H^1(\Omega))} + \|c^h\|_{L^2(J;H^1(\Omega))} + \|D^{1/2}(u^h)\nabla c^h\|_{L^2(\Omega_T)} \le C, \tag{2.14}$$

with constant C independent of h.

Proof. Take $v = p^h - \varphi_2^h \in l_h(V)$ in (2.10) to have

$$\begin{aligned} \|\nabla p^h\|^2_{L^2(\Omega)} + (a_1(c^h)p^h, p^h)_{\Gamma_1^p} &\le C\{\|\rho g\|^2_{L^2(\Omega)} + \|q^I - q^P\|^2_{H^{-1}(\Omega)} \\ &\quad + \|\varphi_1\|^2_{H^{-1/2}(\Gamma_1^p)} + \|\varphi_2^h\|^2_{H^1(\Omega)}\} + \epsilon\|p^h\|^2_{L^2(\Omega)}, \quad t \in J, \end{aligned} \tag{2.15}$$

for any $\epsilon > 0$. Apply a variant of the Poincare inequality

$$\|p^h\|_{L^2(\Omega)} \leq C\{\|\nabla p^h\|_{L^2(\Omega)} + \|p_h\|_{L^2(\Gamma_{1,*}^p)} + \|\varphi_2^h\|_{H^1(\Omega)}\}, \tag{2.16}$$

and the inequality

$$\|\varphi_2^h\|_{H^1(\Omega)} \leq \|\varphi_2\|_{H^1(\Omega)}$$

to obtain the bound for p^h in (2.14).

Now, choose $v = c^h - \varphi_4^h \in l_h(W)$ in (2.11) and use (2.13), Lemma 2.2, and the above bound on p^h to see that

$$\begin{aligned} &\textstyle\int_J(\phi\partial^{-h}c^h, c^h - \varphi_4^h)\mathrm{d}t + C_1\int_J(D(u^h)\nabla c^h, \nabla c^h)\mathrm{d}t \\ &\quad\leq C(T,\Omega)\{1 + \textstyle\int_J\big(\|q^I\|^2_{H^{-1}(\Omega)} + \|\varphi_3\|^2_{H^{-1/2}(\Gamma_1^c)} \\ &\qquad\qquad + \|\varphi_4^h\|^2_{H^1(\Omega)} + \|\varphi_4^h\|^4_{W^{1,4}(\Omega)}\big)\mathrm{d}t\}. \end{aligned} \tag{2.17}$$

Next, it is easy to see that

$$\int_J(\phi\partial^{-h}c^h, c^h)\mathrm{d}t = \sum_{i=1}^m(\phi(c^i - c^{i-1}), c^i) \geq \frac{1}{2}\{(\phi c^m, c^m) - (\phi c^0, c^0)\}. \tag{2.18}$$

Also, we find that

$$\begin{aligned} \textstyle\int_J(\phi\partial^{-h}c^h, \varphi_4^h)\mathrm{d}t &= (\phi c^m, \varphi_4^m) - (\phi c^0, \varphi_4^1) - \textstyle\int_0^{T-h}(\phi c^h, \partial^h\varphi_4^h)\mathrm{d}t \\ &\leq C\{\|\varphi_4^h\|_{L^\infty(J;L^1(\Omega))} + \textstyle\int_0^{T-h}\|\partial^h\varphi_4^h\|_{L^1(\Omega)}\mathrm{d}t\}. \end{aligned} \tag{2.19}$$

Finally, it is obvious that

$$\begin{aligned} &\|\varphi_4^h\|_{L^2(J;W^{1,4}(\Omega))} \leq C\|\varphi_4\|_{L^2(J;W^{1,4}(\Omega))}, \\ &\|\varphi_4^h\|_{L^\infty(J;L^1(\Omega))} \leq C\|\varphi_4\|_{W^{1,1}(J;L^1(\Omega))}, \end{aligned} \tag{2.20}$$

and

$$\begin{aligned} \textstyle\int_0^{T-h}\|\partial^h\varphi_4^h\|_{L^1(\Omega)}\mathrm{d}t &= \textstyle\sum_{i=1}^{n-1}\|\varphi_4^{i+1} - \varphi_4^i\|_{L^1(\Omega)} \\ &= \textstyle\sum_{i=1}^{n-1}\frac{1}{h}\|\int_{t_{i-1}}^{t_i}\int_t^{t+h}\partial_t\varphi_4(\cdot,\tau)\mathrm{d}\tau\mathrm{d}t\|_{L^1(\Omega)} \\ &\leq \textstyle\int_J\|\partial_t\varphi_4\|_{L^1(\Omega)}\mathrm{d}t. \end{aligned} \tag{2.21}$$

Now, combine (2.17)–(2.21) to have the desired result for c^h. []

Corollary 2.5. *For any $2 \leq r < \infty$, for a subsequence $p^h \rightharpoonup p$ weakly in $L^r(J; H^1(\Omega))$ and $c^h \rightharpoonup c$ weakly in $L^2(J; H^1(\Omega))$. Furthermore, $p \in L^\infty(J; V) + \varphi_2$, $c \in L^2(J; W) + \varphi_4$, and*

$$0 \leq c(x,t) \leq 1 \quad a.e.\ on \quad \Omega_T. \tag{2.22}$$

Proof. It follows from Lemma 2.4 that $c^h - \varphi_4^h$ converges weakly in $L^2(J; W)$. Since $\varphi_4^h \rightharpoonup \varphi_4$ weakly in $L^2(J; H^1(\Omega))$, $c^h \rightharpoonup c$ weakly in $L^2(J; H^1(\Omega))$ with $c \in L^2(J; W) + \varphi_4$. The same argument shows that $p^h \rightharpoonup p$ weakly in $L^r(J; H^1(\Omega))$ with $p \in L^r(J; V) + \varphi_2$ for $2 \leq r < \infty$. Since $\|p\|_{L^r(J;H^1(\Omega))} \leq C$ with C independent of r, in fact $p \in L^\infty(J; V) + \varphi_2$. Finally, (2.22) follows from (2.13). []

Lemma 2.6. *There is a subsequence such that $c^h \to c$ strongly in $L^2(\Omega_T)$.*

This lemma will be shown also in the next subsection §2.4.

Corollary 2.7. *There is a subsequence such that $c^h \to c$ strongly in $L^2(J; H^{1-\pi}(\Omega))$ and $L^2(J; H^{1/2-\pi}(\partial\Omega))$ for any $0 < \pi < 1/2$, and $c^h \to c$ pointwisely a.e. on Ω_T.*

Proof. Apply the interpolation inequality

$$\|v\|_{H^\sigma(\Omega)} \leq \delta \|v\|_{H^1(\Omega)} + C_\delta \|v\|_{L^2(\Omega)}, \tag{2.23}$$

for any $0 < \sigma < 1$ and $\delta > 0$, the boundedness of the trace operator, and Lemma 2.6 to prove the desired statement. []

We are now ready to prove Theorem 2.1.

Proof of Theorem 2.1. From Corollaries 2.5 and 2.7, (2.10) implies (2.7) since $\cup_{M=1}^\infty l_h(V)$ is dense in $L^\infty(J;V)$. Also, it follows from (2.11) that

$$\begin{aligned} &\lim_{h\to 0^+} \left\{ \int_J (\phi\partial^{-h} c^h, v)\mathrm{d}t + \int_J (D(u^h)\nabla c^h, \nabla v)\mathrm{d}t + \int_J (u^h \cdot \nabla c^h, v)\mathrm{d}t \right\} \\ &\quad + \int_J (q^I c, v)\mathrm{d}t + \int_J (a_2(c)c, v)_{\Gamma_1^c}\mathrm{d}t \\ &\qquad = \int_J (\hat{c} q^I, v)\mathrm{d}t - \int_J (\varphi_3, v)_{\Gamma_1^c}\mathrm{d}t, \quad \forall v \in \cup_{M=1}^\infty l_h(W(u^h)). \end{aligned} \tag{2.24}$$

By (2.7) and (2.10), we see that

$$\begin{aligned} &(a(c^h)\nabla[p^h - p], \nabla[p^h - p]) + (a(c^h)\nabla[p^h - p], \nabla p) \\ &+([a(c) - a(c^h)](\nabla p - \rho g), \nabla p^h) + (a_1(c^h)(p - p^h), p^h)_{\Gamma_1^p} \\ &+([a_1(c) - a_1(c^h)]p, p^h)_{\Gamma_1^p} = 0, \end{aligned}$$

which shows that $\nabla p^h \to \nabla p$ strongly in $(L^2(\Omega_T))^d$, so $u^h \to u$ strongly in $(L^2(\Omega_T))^d$ and $D(u^h) \to D(u)$ strongly in $(L^2(\Omega_T))^{d\times d}$. Hence, we obtain

$$\begin{aligned} &\lim_{h\to 0^+} \left\{ \int_J (D(u^h)\nabla c^h, \nabla v)\mathrm{d}t + \int_J (u^h \cdot \nabla c^h, v)\mathrm{d}t \right\} \\ &\quad = \int_J (D(u)\nabla c, \nabla v)\mathrm{d}t + \int_J (u \cdot \nabla c, v)\mathrm{d}t. \end{aligned}$$

Next, for any $v \in L^2(J; W(u))$, $v^h \in l_h(W(u^h))$ for h sufficiently small, where $v^h(x,t) = h^{-1}\int_{J_k} v(x,\tau)\mathrm{d}\tau$. Then, by (2.11), (2.14), and the compact embedding relation $H^1(\Omega) \hookrightarrow L^4(\Omega)$, we observe that

$$\int_J (\phi\partial^{-h} c^h, v)\mathrm{d}t = \int_J (\phi\partial^{-h} c^h, v^h)\mathrm{d}t \leq C\left\{ \|D^{1/2}(u)\nabla v\|_{L^2(\Omega_T)} + \|v\|_{L^2(\Omega_T)} \right\},$$

for h sufficiently small. Consequently, for a subsequence $\phi\partial^{-h} c^h$ converges weakly in $L^2(J; W^*(u))$. If $v \in C_0^\infty(\Omega_T)$, with $h > 0$ small enough we see that

$$\int_J (\phi\partial^{-h} c^h, v)\mathrm{d}t = -\int_0^{T-h} (\phi c^h, \partial^h v)\mathrm{d}t \to -\int_J (\phi c, \partial_t v)\mathrm{d}t = \int_J \langle \phi\partial_t c, v\rangle\, \mathrm{d}t,$$

as a distribution. Therefore, $\phi\partial^{-h} c^h \rightharpoonup \phi\partial_t c$ weakly in $L^2(J; W^*(u))$. Combining these results, (2.8) follows from (2.24) since $\cup_{M=1}^\infty l_h(W(u^h))$ is dense in $L^2(J; W(u))$. Also, as for (2.17), it can be shown that $c \in L^2(J; W(u))$ from (2.8).

Finally, if $v \in L^2(J; W(u)) \cap W^{1,1}(J; L^1(\Omega))$ with $v(x, T) = 0$, we find that

$$\int_J (\phi\partial^{-h} c^h, v)\mathrm{d}t + \int_0^{T-h} (\phi[c^h - c^0], \partial^h v)\mathrm{d}t = \frac{1}{h}\int_{T-h}^T (\phi[c^h - c^0], v)\mathrm{d}t,$$

which yields (2.9), and thus the proof of Theorem 2.1 is complete. []

2.4. Proof of Lemmas 2.3 and 2.6. In this subsection the possibility that c is outside $[0,1]$ is allowed. All functions of c are extended constantly outside $[0,1]$.

Lemma 2.3 is purely an elliptic result, and will obviously follow from the next proposition. For notational convenience the superscript h is omitted below.

Proposition 2.8. *In addition to assumptions* (A1)–(A10)*, suppose that* $0 < \eta_* \leq \eta_1(x) \in L^\infty(\Omega)$ *and* $0 \leq \eta_2(x) \leq \eta_1(x)$*. Then, for* η_* *sufficiently big, the following problem has a weak solution* $(p,c) \in (V+\varphi_2)\times(W+\varphi_4)$*:*

$$\begin{aligned}(a(c)\{\nabla p - \rho g\}, \nabla v) + (a_1(c)p, v)_{\Gamma_1^p} \\ = (q^I - q^P, v) - (\varphi_1, v)_{\Gamma_1^p}, \quad \forall v \in V,\end{aligned} \tag{2.25}$$

$$\begin{aligned}(\eta_1 c, v) + (D(u)\nabla c, \nabla v) + (u\cdot\nabla c, v) + (q^I c, v) + (a_2(c)c, v)_{\Gamma_1^c} \\ = (\hat{c}q^I, v) - (\varphi_3, v)_{\Gamma_1^c} + (\eta_2, v), \quad \forall v \in W,\end{aligned} \tag{2.26}$$

and

$$0 \leq c(x,t) \leq 1 \qquad a.e.\ on\ \Omega_T, \tag{2.27}$$

where u *is given as in* (2.12).

Proof. Let $\{v_i^1\}_{i=1}^\infty$ and $\{v_i^2\}_{i=1}^\infty$ be bases for V and W, respectively, and set V_m =span$\{v_1^1,\ldots,v_m^1\}$ and W_m =span$\{v_1^2,\ldots,v_m^2\}$. With V_m and W_m replacing V and W in (2.25) and (2.26), respectively, we obtain a Galerkin procedure.

For $v^j = \sum_{i=1}^m \beta_i^j v_i^j$, $j=1,2$, we introduce the mapping $\Phi_m : \Re^{2m} \to \Re^{2m}$ by

$$\Phi_m \begin{pmatrix} \beta^1 \\ \beta^2 \end{pmatrix} = \begin{pmatrix} \hat\beta^1 \\ \hat\beta^2 \end{pmatrix},$$

where

$$\begin{aligned}\hat\beta_i^1 =& (a(v^2+\varphi_4)\{\nabla(v^1+\varphi_2) - \rho g\}, \nabla v_i^1) + (a_1(v^2+\varphi_4)(v^1+\varphi_2), v_i^1)_{\Gamma_1^p} \\ & -(q^I - q^P, v_i^1) + (\varphi_1, v_i^1)_{\Gamma_1^p}, \\ \hat\beta_i^2 =& (\eta_1(v^2+\varphi_4), v_i^2) + (D(\hat u)\nabla(v^2+\varphi_4), \nabla v_i^2) + (\hat u\cdot\nabla(v^2+\varphi_4), v_i^2) \\ & +(q^I(v^2+\varphi_4), v_i^2) + (a_2(v^2+\varphi_4)(v^2+\varphi_4), v_i^2)_{\Gamma_1^c} \\ & -(\hat c q^I, v_i^2) + (\varphi_3, v_i^2)_{\Gamma_1^c} - (\eta_2, v_i^2),\end{aligned}$$

with $u = -a(v^2+\varphi_4)\{\nabla(v^1+\varphi_2) - \rho g\}$ and $\hat u = mu/(m+|u|)$. By the assumptions (A1)–(A10), Φ_m is continuous. Also, it can be easily seen that

$$\begin{aligned}\Phi_m \begin{pmatrix} \beta^1 \\ \beta^2 \end{pmatrix} \cdot \begin{pmatrix} \beta^1 \\ \beta^2 \end{pmatrix} \geq & C_1(m)\{\|\nabla v^1\|^2_{L^2(\Omega)} + \|\nabla v^2\|^2_{L^2(\Omega)}\} - \epsilon\|v^1\|^2_{L^2(\Omega)} \\ & +(\eta_1(v^2+\varphi_4) - \eta_2, v^2) + (q^I\varphi_4, v^2) \\ & -C\{\|\varphi_1\|^2_{H^{-1}(\Gamma_1^p)} + \|\varphi_3\|^2_{H^{-1}(\Gamma_1^c)} + \|\varphi_2\|^2_{H^1(\Omega)} \\ & \quad +\|\varphi_4\|^2_{H^1(\Omega)} + \|q^I\|^2_{H^{-1}(\Omega)} + \|q^P\|^2_{H^{-1}(\Omega)} \\ & \quad +\|\rho g\|^2_{L^2(\Omega)} + \|v^2\|^2_{L^2(\Omega)}\},\end{aligned}$$

for any $\epsilon > 0$. Note that

$$(\eta_1(v^2+\varphi_4) - \eta_2, v^2) \geq \frac{1}{2}\eta_*\|v^2\|^2_{L^2(\Omega)} - C\{1 + \|\varphi_4\|^2_{L^2(\Omega)}\},$$

and, by the compact embedding relation $H^1(\Omega) \hookrightarrow L^4(\Omega)$ again,

$$\begin{aligned}(q^I\varphi_4, v^2) &\leq \|q^I\|_{L^2(\Omega)}\|\varphi_4\|_{L^4(\Omega)}\|v^2\|_{L^4(\Omega)}\\ &\leq C\|q^I\|_{L^2(\Omega)}\|\varphi_4\|_{L^4(\Omega)}\|v^2\|_{H^1(\Omega)}.\end{aligned}$$

Now, with Poincare's inequality and η_* big enough, combining these results yields that

$$\Phi_m\begin{pmatrix}\beta^1\\ \beta^2\end{pmatrix}\cdot\begin{pmatrix}\beta^1\\ \beta^2\end{pmatrix} \geq C_1(m)\big\{\|v^1\|^2_{H^1(\Omega)}+\|v^2\|^2_{H^1(\Omega)}\big\}-C,$$

which is strictly positive for $|\beta^1|+|\beta^2|$ sufficiently big. As a result, Φ_m has a zero; i.e., there is a solution to the Galerkin approximation with $\hat{u}$ replacing u for each m.

As in the proof of Lemma 2.4, it can be seen that this modified Galerkin solutions p^m and c^m are uniformly bounded in $H^1(\Omega)$ (independently of m), so for a subsequence $p^m \rightharpoonup p$ and $c^m \rightharpoonup c$ weakly in $H^1(\Omega)$ with $p \in V+\varphi_2$ and $c \in W+\varphi_4$. Moreover, $c_m \to c$ strongly in $H^{1-\pi}(\Omega)$ $(0<\pi<1/2)$ and pointwisely a. e. both on Ω and $\partial\Omega$. Therefore, (p,c) is a weak solution to the system in (2.25) and (2.26). Finally, a standard maximum principle argument on (2.26) (with $v=\min(c,0)$ and $v=\max(c-1,0)\in W$, respectively, in (2.26)) can be applied to show (2.27). This completes the proof of the proposition. $\Box$

The next lemma is related to Lemma 3 in [6], and needed for proving Lemma 2.6.

Lemma 2.9. *Let c^h satisfy* (2.11). *Then there exists C such that, for any $\zeta>0$,*

$$\frac{1}{\zeta}\int_\zeta^T \|\phi^{1/2}\big(c^h(\cdot,t)-c^h(\cdot,t-\zeta)\big)\|^2_{L^2(\Omega)}\mathrm{d}t \leq C.$$

Proof. Let k be fixed $(1\leq k\leq m)$; for $\tau \in J_i$, we define the interval $Q=Q(\tau)=((i-k)h, ih]$, and the characteristic function χ_Q. Take $v(x,t)=kh\chi_Q(t)\partial^{-kh}(c^h-\varphi_4^h)(x,\tau)\in l_h(W)$ in (2.11), and apply the relation

$$\int_J \partial^{-h}c^h\chi_Q\mathrm{d}t=\sum_{j=i-k+1}^{i}(c^j-c^{j-1})=kh\partial^{-kh}c^h(\cdot,\tau),$$

(2.13), and (2.14) to obtain

$$kh\int_{kh}^T \|\phi^{1/2}\partial^{-kh}c^h(\cdot,\tau)\|^2_{L^2(\Omega)}\mathrm{d}\tau \leq C+kh\int_{kh}^T(\phi\partial^{-kh}c^h(\cdot,\tau),\partial^{-kh}\varphi_4^h(\cdot,\tau))\mathrm{d}\tau.$$

As for (2.21), it can be shown that

$$\int_{kh}^T\|\partial^{-kh}\varphi_4^h\|_{L^1(\Omega)}\mathrm{d}\tau \leq \|\varphi_4\|_{L^1(\Omega_T)}.$$

Now, combine these two results to have the desired result since c^h is constant on each interval J_i. $\Box$

For m_1 a positive integer, let $\delta=T/m_1$ and $J_k^\delta=((k-1)\delta,k\delta]$. Introduce the operator $A^\delta: L^1(J)\to L^1(J)$ by

$$A^\delta(v)=\frac{1}{\delta}\int_{J_k^\delta}v(\tau)\mathrm{d}\tau, \quad t\in J_k^\delta.$$

We are now in a position to prove Lemma 2.6.

PROOF OF LEMMA 2.6. For any ζ, $N > 0$, define

$$Q = Q(c^h, \zeta, N) = \big\{t \in (\zeta, T] : \|c^h(\cdot, t)\|^2_{H^1(\Omega)} + \|c^h(\cdot, t-\zeta)\|^2_{H^1(\Omega)} \\ + \tfrac{1}{\zeta}\|\phi^{1/2}\big(c^h(\cdot, t) - c^h(\cdot, t-\zeta)\big)\|^2_{L^2(\Omega)} > N\big\}.$$

Obviously, by (2.13), (2.14), and Lemma 2.9, the measure of Q is less than C/N with constant C independent of h. If $t \in (\zeta, T] \setminus Q$, then

$$\|c^h(\cdot, t) - c^h(\cdot, t-\zeta)\|^2_{L^2(\Omega)} \leq \frac{\zeta N}{\phi_*}.$$

Thus we see that

$$\int_\zeta^T \|c^h(\cdot, t) - c^h(\cdot, t-\zeta)\|^2_{L^2(\Omega)} \mathrm{d}t \leq \frac{\zeta N T}{\phi_*} + \frac{4C|\Omega|}{N},$$

so that, by the arbitraryness of N,

$$\int_\zeta^T \|c^h(\cdot, t) - c^h(\cdot, t-\zeta)\|^2_{L^2(\Omega)} \mathrm{d}t \to 0 \qquad \text{as } \zeta \to 0^+,$$

uniformly in h. Therefore, by the definition of A^δ we see that

$$\int_J \|c^h - A^\delta(c^h)\|^2_{L^2(\Omega)} \mathrm{d}t \leq \tfrac{2}{\delta} \int_0^\delta \int_\zeta^T \|c^h(\cdot, t) - c^h(\cdot, t-\zeta)\|^2_{L^2(\Omega)} \mathrm{d}t\mathrm{d}\zeta \\ \to 0 \qquad \text{as } \delta \to 0^+, \tag{2.28}$$

uniformly in h. Also, $\|A^\delta(c^h)\|_{L^2(J, H^1(\Omega))}$ is uniformly bounded, so for fixed $\delta > 0$, $A^\delta(c^h)$ converges strongly in $L^2(\Omega_T)$ as $h \to 0^+$. Consequently, apply (2.28) and the inequality

$$\|c^{h_1} - c^{h_2}\|_{L^2(\Omega_T)} \leq \sum_{j=1}^2 \|c^{h_j} - A^\delta(c^{h_j})\|_{L^2(\Omega_T)} + \|A^\delta(c^{h_1}) - A^\delta(c^{h_2})\|_{L^2(\Omega_T)},$$

to complete the proof of Lemma 2.6. $\square$

3. Two-Phase Flow and Transport. In this section we consider two-phase flow and transport equations in a porous medium $\Omega \subset \Re^d$ ($d \leq 3$). These equations will be reviewed in §3.1. The differential system we shall study in this section will be derived in §3.2. The main result in this section is stated in §3.3.

3.1. Flow and transport equations. The mass balance equation for each of the fluid phases is given by ([7])

$$\phi \frac{\partial(\rho_\alpha s_\alpha)}{\partial t} + \nabla \cdot (\rho_\alpha u_\alpha) = \rho_\alpha q_\alpha, \quad \alpha = w, n, \tag{3.1}$$

where $\alpha = w$ denotes the wetting phase (e.g. water), $\alpha = n$ indicates the nonwetting phase (e.g. oil or air), ϕ is the porosity of the porous medium, and ρ_α, s_α, u_α, and q_α are, respectively, the density, (reduced) saturation, volumetric velocity, and external volumetric flow rate of the α-phase. The volumetric velocity u_α is given again by the Darcy law

$$u_\alpha = -\frac{kk_{r\alpha}}{\mu_\alpha}(\nabla p_\alpha - \rho_\alpha g), \quad \alpha = w, n, \tag{3.2}$$

where k is the absolute permeability of the porous medium, and p_α, μ_α, and $k_{r\alpha}$ are the pressure, viscosity, and relative permeability of the α-phase, respectively. In addition to (3.1) and (3.2), the customary property for the saturations is

$$s_w + s_n = 1, \tag{3.3}$$

and the two pressures are related by the capillary pressure function:

$$p_c = p_n - p_w. \tag{3.4}$$

With the flow of the fluids specified as above, an equation for the transport of a chemical constituent is needed. The constituent can be transported in each of the phases, so the equation is written for each phase. Let c_α denote the mass concentration of the constituent in the α phase. Then the mass balance law for the constituent in each phase reads as follows:

$$\phi\frac{\partial(\rho_\alpha s_\alpha c_\alpha)}{\partial t} + \nabla\cdot(\rho_\alpha u_\alpha c_\alpha - \phi\rho_\alpha s_\alpha D_\alpha \nabla c_\alpha) + \phi r_\alpha \rho_\alpha s_\alpha c_\alpha = \hat{c}_\alpha \rho_\alpha q_\alpha, \tag{3.5}$$

for $\alpha = w$, n, where D_α, r_α, and $\hat{c}_\alpha$ are the diffusion/dispersion coefficient, reaction rate, and concentration in the external flow for the α phase, respectively.

3.2. The differential system. The following functional dependence is physically reasonable:

$$k_{r\alpha} = k_{r\alpha}(x, s_\alpha), \quad p_c = p_c(x, s_w), \quad D_\alpha = D_\alpha(x, u_\alpha).$$

However, to extend the analysis of the last section to the present problem we assume that the viscosity μ_α is independent of c_α. The case that $\mu_\alpha = \mu_\alpha(c_\alpha)$ needs to be handled with a different argument, and will be treated in a forthcoming paper. The model derived here is of interest in itself.

In order to separate the pressure and saturation equations, we introduce the phase mobility functions

$$\lambda_\alpha(x, s_\alpha) = k_{r\alpha}/\mu_\alpha, \quad \alpha = w, n,$$

and the total mobility

$$\lambda(x, s) = \lambda_w + \lambda_n,$$

where $s = s_w$. The fractional flow functions are defined by

$$f_\alpha(x, s) = \lambda_\alpha/\lambda, \quad \alpha = w, n.$$

Following [4, 8], we define the global pressure

$$p = p_n - \int_0^s \left(f_w \frac{\partial p_c}{\partial s}\right)(x, \xi)\mathrm{d}\xi. \tag{3.6}$$

Also, we shall use the complementary pressure [6]

$$\theta = -\int_0^s \big(f_w f_n \frac{\partial p_c}{\partial s}\big)(x,\xi)\mathrm{d}\xi. \tag{3.7}$$

Finally, we define the total velocity

$$u = u_w + u_n. \tag{3.8}$$

Now, under the assumption that the fluids are incompressible we apply (3.3) and (3.8) to (3.1) to see that

$$\nabla \cdot u = q \equiv q_w + q_n, \tag{3.9}$$

and (3.4), (3.6), and (3.8) to (3.2) to obtain

$$u = -k(\lambda \nabla p + \gamma_1), \tag{3.10}$$

where

$$\gamma_1 = -\lambda_w \nabla_x p_c + \lambda \int_0^s \nabla_x \big(f_w \frac{\partial p_c}{\partial s}\big)(x,\xi)\mathrm{d}\xi - \big(\lambda_w \rho_w + \lambda_n \rho_n\big) g.$$

Similarly, apply (3.4), (3.6), and (3.7) to (3.1) and (3.2) with $\alpha = w$ to have

$$\phi \frac{\partial s}{\partial t} - \nabla \cdot \big\{k\big(\lambda \nabla \theta + \lambda_w \nabla p + \gamma_2\big)\big\} = q_w, \tag{3.11}$$

where

$$\gamma_2 = -\lambda_w \nabla_x p_c + \lambda_w \int_0^s \nabla_x \big(f_w \frac{\partial p_c}{\partial s}\big)(x,\xi)\mathrm{d}\xi + \lambda \int_0^s \nabla_x \big(f_w f_n \frac{\partial p_c}{\partial s}\big)(x,\xi)\mathrm{d}\xi - \lambda_w \rho_w g.$$

Finally, it can be seen that the phase velocities are determined by

$$\begin{aligned} u_w &= -k\big(\lambda \nabla \theta + \lambda_w \nabla p + \gamma_2\big), \\ u_n &= k\big(\lambda \nabla \theta - \lambda_n \nabla p + \gamma_3\big), \end{aligned} \tag{3.12}$$

where

$$\gamma_3 = -\lambda_n \int_0^s \nabla_x \big(f_w \frac{\partial p_c}{\partial s}\big)(x,\xi)\mathrm{d}\xi + \lambda \int_0^s \nabla_x \big(f_w f_n \frac{\partial p_c}{\partial s}\big)(x,\xi)\mathrm{d}\xi + \lambda_n \rho_n g.$$

The pressure equation is given by (3.9) and (3.10), while the saturation equation is described by (3.11). In hydrology, it is common to replace the pressures by the pressure heads

$$h_\alpha = p_\alpha / (\rho_{sw} g), \quad \alpha = w, n,$$

where ρ_{sw} is the density of water at the standard temperature and pressure. With the pressure heads h_α, similar equations to those in (3.9)–(3.12) can be obtained; in this paper we shall use the pressures.

To derive the concentration equation, we make use of the usual equilibrium assumption on mass transfer. That is, the constituent instantaneously establishes an equilibrium distribution between the two phases. Then the concentration in each phase is proportional to that in the other phase:

$$c_n = Hc_w,$$

where H is called the Henry constant and taken to be one for simplicity. Then under the incompressibility assumption, apply (3.3) and (3.8) to (3.5) to find that

$$\phi \frac{\partial c}{\partial t} - \nabla \cdot \big(D\nabla c - uc\big) + Rc = \hat{c}q, \tag{3.13}$$

where $c = c_w$, $D = \phi(s_w D_w + s_n D_n)$, and $R = \phi(s_w r_w + s_n r_n)$.

In summary, from (3.9)–(3.11) and (3.13) we have the differential system

$$\begin{aligned}
&-\nabla \cdot \big\{k(\lambda(s,c)\nabla p + \gamma_1(s,c))\big\} = q(s,c),\\
&\phi\partial_t s - \nabla \cdot \big\{k\big(\lambda(s,c)\nabla\theta + \lambda_w(s,c)\nabla p + \gamma_2(s,c)\big)\big\} = q_w(s,c),\\
&\phi\partial_t c - \nabla \cdot \big\{D(s,u_w,u_n)\nabla c - uc\big\} + R(s)c = \hat{c}q(s,c),
\end{aligned} \tag{3.14}$$

where

$$\begin{aligned}
u &= -k(\lambda(s,c)\nabla p + \gamma_1(s,c)),\\
u_w &= -k\big(\lambda(s,c)\nabla\theta + \lambda_w(s,c)\nabla p + \gamma_2(s,c)\big),\\
u_n &= k\big(\lambda(s,c)\nabla\theta - \lambda_n(s,c)\nabla p + \gamma_3(s,c)\big).
\end{aligned} \tag{3.15}$$

Finally, s is related to θ through (3.7):

$$s = \mathcal{S}(\theta), \tag{3.16}$$

where $\mathcal{S}(x,\theta)$ is the inverse of (3.7) for $0 \le \theta \le \theta^*(x)$ with

$$\theta^*(x) = -\int_0^1 \big(f_w f_n \frac{\partial p_c}{\partial s}\big)(x,\xi)\mathrm{d}\xi.$$

The differential system in (3.14)–(3.16) determines the main unknowns p, s, θ, and c. The model is completed by specifying boundary and initial conditions.

With the following division of Γ:

$$\begin{aligned}
\Gamma &= \Gamma_1^p \cup \Gamma_2^p = \Gamma_1^\theta \cup \Gamma_2^\theta = \Gamma_1^c \cup \Gamma_2^c,\\
\emptyset &= \Gamma_1^p \cap \Gamma_2^p = \Gamma_1^\theta \cap \Gamma_2^\theta = \Gamma_1^c \cap \Gamma_2^c.
\end{aligned}$$

The boundary conditions are specified by

$$\begin{aligned}
&u\cdot\nu - a_1(s,c)p = \varphi_1(x,t), && (x,t) \in \Gamma_1^p \times J,\\
&p = \varphi_2(x,t), && (x,t) \in \Gamma_2^p \times J,\\
&u_w\cdot\nu - a_2(s,c)\theta = \varphi_3(x,t), && (x,t) \in \Gamma_1^\theta \times J,\\
&\theta = \varphi_4(x,t), && (x,t) \in \Gamma_2^\theta \times J,\\
&(uc - D\nabla c)\cdot\nu - a_3(s,c)c = \varphi_5(x,t), && (x,t) \in \Gamma_1^c \times J,\\
&c = \varphi_6(x,t), && (x,t) \in \Gamma_2^c \times J,
\end{aligned} \tag{3.17}$$

where the a_i and φ_i are given functions. The initial conditions are given by

$$\begin{aligned}
\theta(x,0) &= \theta_0(x), \quad x \in \Omega,\\
c(x,0) &= c_0(x), \quad x \in \Omega.
\end{aligned} \tag{3.18}$$

The differential system has a clear structure. Note that while λ_w and λ_n can be zero, λ is always positive (see the assumptions below). That is, the pressure equation is elliptic, and the saturation and concentration equations are parabolic. This model has been analyzed from the computational point of view using finite elements in [9, 10, 11].

3.3. The main result. The assumptions on the physical data are stated below; some of the assumptions in the previous section are repeated for completedness.

(B1) Assume that $\Omega \subset \Re^d$ is a multiply-connected domain with Lipschitz boundary Γ, $\Gamma = \Gamma_1^p \cup \Gamma_2^p = \Gamma_1^\theta \cup \Gamma_2^\theta = \Gamma_1^c \cup \Gamma_2^c$, $\Gamma_1^p \cap \Gamma_2^p = \Gamma_1^\theta \cap \Gamma_2^\theta = \Gamma_1^c \cap \Gamma_2^c = \emptyset$, and each Γ_i^p, Γ_i^θ, and Γ_i^c is a $(d-1)$-dimensional domain.

(B2) Assume that $\phi \in L^\infty(\Omega)$ and $\phi(x) \geq \phi_* > 0$, and that $k(x)$ is a bounded, symmetric, and uniformly positive definite matrix, i.e.,

$$0 < k_* \leq |\xi|^{-2} \sum_{i,j=1}^{d} k_{ij}(x)\xi_i\xi_j \leq k^* < \infty, \quad x \in \Omega,\ \xi \neq 0 \in \Re^d.$$

(B3) The diffusion/dispersion term in (3.5) is assumed to be Fickian in form with the coefficient given by ([7])

$$D_\alpha(u_\alpha) = d_{m_\alpha} I + |u_\alpha|\big(d_{l_\alpha} E(u_\alpha) + d_{t_\alpha} E^-(u_\alpha)\big), \quad \alpha = w, n,$$

where $d_{m_\alpha} > 0$ is the molecular diffusion coefficient, d_{l_α} and d_{t_α} are the longitudinal and transverse dispersion coefficients, respectively, for the α phase, the matrix $E(u_\alpha)$ is the projection along the direction of flow determined by

$$E(u_\alpha) = \left(\frac{u_{\alpha,i}u_{\alpha,j}}{|u_\alpha|^2}\right), \quad |u_\alpha| = \sqrt{u_{\alpha,1}^2 + u_{\alpha,2}^2}, \quad u_\alpha = (u_{\alpha,1}, u_{\alpha,2}),$$

and $E^-(u_\alpha) = I - E(u_\alpha)$.

(B4) $\lambda_\alpha(x,s,c)$ is measurable in $x \in \Omega$ and continuous in s, $c \in [0,1]$, and satisfies that $0 \leq \lambda_\alpha \leq \lambda_\alpha^* < \infty$. Assume that $0 < \lambda_* \leq \lambda(x,s,c) \leq \lambda^* < \infty$, $x \in \Omega$, s, $c \in [0,1]$.

(B5) Assume that $0 < \theta^* \in H^1(\Omega)$ and that $\mathcal{S} : \{(x,\theta) : x \in \Omega, 0 \leq \theta \leq \theta^*(x)\} \to [0,1]$ is measurable in x, continuous and strictly monotone increasing in θ, and satisfies that $\mathcal{S}(x,0) = 0$ and $\mathcal{S}(x,\theta^*(x)) = 1$.

(B6) Suppose that γ_1, γ_2, γ_3, q, and q_w are continuous in s and c, and the following norms are bounded:

$$\begin{array}{llll} |||\gamma_1|||_{L^\infty(J;L^2(\Omega))}, & |||\gamma_2|||_{L^2(\Omega_T)}, & \|\varphi_1\|_{L^\infty(J;H^{-1/2}(\Gamma_1^p))}, & |||q|||_{L^\infty(J;L^2(\Omega))}, \\ |||q_w|||_{L^2(J;H^{-1}(\Omega))}, & |||\gamma_3|||_{L^2(\Omega_T)}, & \|\varphi_3\|_{L^2(J;H^{-1/2}(\Gamma_1^\theta))}, & \|\varphi_5\|_{L^2(J;H^{-1/2}(\Gamma_1^c))}, \end{array}$$

where for $v = v(x,s,c)$,

$$|||v||| = \Big\| \sup_{s,c \in [0,1]} |v(x,s,c)| \Big\|,$$

for any given norm.

(B7) Assume that $\partial_t\varphi_4$, $\partial_t\varphi_6 \in L^1(\Omega_T)$ and

$$\varphi_2 \in L^\infty(J;H^1(\Omega)),\ \varphi_4 \in L^2(J;H^1(\Omega)),\ \varphi_6 \in L^2(J;W^{1,4}(\Omega)),$$
$$0 \le \varphi_4(x,t) \le \theta^*(x),\ 0 \le \varphi_6(x,t) \le 1 \quad \text{a.e. on } \Omega_T.$$

(B8) In the case of $\Gamma_2^p = \emptyset$ and $a_1 \equiv 0$, q is independent of s and c, and

$$\int_{\Gamma_1^p} \varphi_1 \mathrm{d}\sigma = \int_\Omega q \mathrm{d}x.$$

(B9) There is a subset $\Gamma_{1,*}^p \subset \Gamma_1^p$ (with nonzero measure only if $\Gamma_2^p = \emptyset$ and $a_1 \not\equiv 0$) such that $a_1 \ge a_{1,*} > 0$ on $\Gamma_{1,*}^p \times J$.

(B10) $a_i \ge 0$ and the norm $|||a_i|||_{L^\infty(\Omega_T)}$ is bounded, $i = 1,2,3$.

(B11) Assume that $q_w(0) \ge 0$ and $q_n(1) = q(1) - q_w(1) \ge 0$ in Ω_T, and that $0 \le \hat{c} \le 1$ a.e. on Ω_T.

(B12) Let θ_0, $c_0 \in L^2(\Omega)$ satisfy $0 \le \theta_0 \le \theta^*(x)$ and $0 \le c_0 \le 1$ a.e. on Ω.

Define the spaces

$$V = \{v \in H^1(\Omega) : v|_{\Gamma_2^p} = 0;\ \text{if } \Gamma_2^p = \emptyset \text{ and } a_1 \equiv 0,\ \text{then } \textstyle\int_\Omega v \mathrm{d}x = 0\},$$
$$W = \{v \in H^1(\Omega) : v|_{\Gamma_2^\theta} = 0\},$$
$$\Lambda = \{v \in H^1(\Omega) : v|_{\Gamma_2^c} = 0\}.$$

Definition 3.1. A weak solution of the system in (3.14)–(3.18) is a triple of functions (p,θ,c) with $p \in L^\infty(J;V)+\varphi_2$, $\theta \in L^2(J;W)+\varphi_4$, $c \in L^2(J;\Lambda(s,u_w,u_n))+\varphi_6$ such that

$$\phi\partial_t s \in L^2(J;W^*), \quad \phi\partial_t c \in L^2(J;\Lambda^*(s,u_w,u_n)),$$

$$0 \le \theta(x,t) \le \theta^*(x), \quad 0 \le c(x,t) \le 1 \quad \text{a.e. on } \Omega_T,$$

$$s = \mathcal{S}(\theta),$$

$$\begin{aligned}&(k\{\lambda(s,c)\nabla p + \gamma_1(s,c)\}, \nabla v) + (a_1(s,c)p, v)_{\Gamma_1^p} \\ &\quad = (q(s,c), v) - (\varphi_1, v)_{\Gamma_1^p}, \quad \forall v \in L^\infty(J;V),\end{aligned}$$

$$\begin{aligned}&\textstyle\int_J \langle \phi\partial_t s, v\rangle \,\mathrm{d}t + \int_J (k\{\lambda(s,c)\nabla\theta + \lambda_w(s,c)\nabla p + \gamma_2(s,c)\}, \nabla v)\mathrm{d}t \\ &\quad + \textstyle\int_J (a_2(s,c)\theta, v)_{\Gamma_1^\theta}\mathrm{d}t = \int_J (q_w(s,c), v)\mathrm{d}t - \int_J (\varphi_3, v)_{\Gamma_1^\theta}\mathrm{d}t, \quad \forall v \in L^2(J;W),\end{aligned}$$

$$\begin{aligned}&\textstyle\int_J \langle \phi\partial_t s, v\rangle \,\mathrm{d}t + \int_J (\phi(s-s_0), \partial_t v)\mathrm{d}t = 0, \\ &\qquad \forall v \in L^2(J;W) \cap W^{1,1}(J;L^1(\Omega)), \quad v(x,T) = 0,\end{aligned}$$

$$\begin{aligned}&\textstyle\int_J \langle \phi\partial_t c, v\rangle \,\mathrm{d}t + \int_J (D(s,u_w,u_n)\nabla c - uc, \nabla v)\mathrm{d}t + \int_J (R(s)c, v)\mathrm{d}t \\ &\quad + \textstyle\int_J (a_3(s,c)c, v)_{\Gamma_1^c}\mathrm{d}t = \int_J (\hat{c}q(s,c), v)\mathrm{d}t - \int_J (\varphi_5, v)_{\Gamma_1^c}\mathrm{d}t, \\ &\qquad\qquad \forall v \in L^2(J;\Lambda(s,u_w,u_n)),\end{aligned}$$

$$\int_J \langle \phi\partial_t c, v\rangle \, dt + \int_J (\phi(c-c_0), \partial_t v) dt = 0,$$
$$\forall v \in L^2(J; \Lambda(s, u_w, u_n)) \cap W^{1,1}(J; L^1(\Omega)), \quad v(x,T) = 0,$$

where $s_0 = \mathcal{S}(\theta_0)$, u, u_w, and u_n are given in (3.15), and

$$\Lambda(s, u_w, u_n) = \{v \in \Lambda : (D(s, u_w, u_n)\nabla v, \nabla v) < \infty\}.$$

Theorem 3.1. *Under assumptions* (B1)–(B12), *the system in* (3.14)–(3.18) *has a weak solution in the sense of Definition* 3.1.

This theorem can be shown as in the last section. The pressure equation can be handled in the same way as for (2.7), and the saturation and concentration equations can be treated with the same manner as for (2.8) and (2.9).

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